Exercise 20

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x\to 10}\left(3-\frac{4}{5}x\right)=-5$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x-10|<\delta \qquad \text{then} \qquad \left|\left(3-\frac{4}{5}x\right)-(-5)\right|<\varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x-10|$.

$$\left|\left(3-\frac{4}{5}x\right)-(-5)\right|<\varepsilon$$

$$\left|8-\frac{4}{5}x\right|<\varepsilon$$

$$\left|-\frac{4}{5}(x-10)\right|<\varepsilon$$

$$\frac{4}{5}|x-10|<\varepsilon$$

$$|x-10|<\frac{5\varepsilon}{4}$$

Choose $\delta = 5\varepsilon/4$. Now, assuming that $|x-10|<\delta$,

$$\begin{aligned}\left|\left(3-\frac{4}{5}x\right)-(-5)\right| &= \left|8-\frac{4}{5}x\right| \\ &= \left|-\frac{4}{5}(x-10)\right| \\ &= \frac{4}{5}|x-10| \\ &< \frac{4}{5}\delta \\ &= \frac{4}{5}\left(\frac{5\varepsilon}{4}\right) \\ &= \varepsilon.\end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 10}\left(3-\frac{4}{5}x\right)=-5.$$